Addendum to `Remarks on the 20-vertex model'

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ADDENDUM

Addendum to 'Remarks on the 20-vertex model'†

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Abstract. In the previous paper, we solved a special case of the reversal-symmetric 20-vertex model on a triangular lattice where the vertex configurations with all arrows reversed are identified. In this paper we do not assume reversal symmetry and obtain a more general result.

The 20-vertex model on a triangular lattice is defined as follows. Place arrows on the bonds of a triangular lattice so that there are three arrows entering and three arrows leaving each vertex. There are 20 possible vertex configurations which are divided into two groups. The first group consists of ten vertex configurations as shown in figure 1 of the previous paper (Lin 1978). These vertices are associated with the energies e_i $(i = 1, \ldots, 10)$. The second group consists of ten vertex configurations associated with the energies e_i' $(i = 1, \ldots, 10)$ such that the configuration corresponding to e_i' is obtained from the configuration corresponding to e_i by reversing all arrows. The partition function is

$$Z = \sum \left(\prod \omega_i^{n_i} \omega_i'^{n_i'} \right) \tag{1}$$

where $\omega_i = \exp(-e_i/kT)$ and $\omega_i' = \exp(-e_i'/kT)$ are the Boltzmann weights (vertex weights), k is the Boltzmann constant, T is the temperature, n_i (n_i') is the number of vertices with energy e_i (e_i'), and the summation is extended to all allowed arrow configurations.

The reversal-symmetric 20-vertex model $(e_i = e_i')$ can be solved by the method of Bethe ansatz if certain conditions among the Boltzmann weights are satisfied (Baxter 1969, Kelland 1974). The general 20-vertex model can be solved by the Pfaffian method if the Boltzmann weights satisfy the free-fermion conditions (Sacco and Wu 1975). In the previous paper (Lin 1978), we solved another special case of the reversal-symmetric 20-vertex model where

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega \qquad \omega_5 = \omega_6 = \omega_7 = \omega_8 = \bar{\omega} \qquad \omega_9 = \omega_{10} = 0 \tag{2}$$

and the free energy per vertex is

$$F = -\frac{kT}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln(2\bar{\omega} + 2\omega \cos\alpha \, e^{i\beta}) \, d\alpha \, d\beta. \tag{3}$$

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The purpose of this paper is to generalise the above result. We shall consider the following special case of the general 20-vertex model:

$$\begin{aligned}
&\omega_1 = \omega_1' \qquad \omega_2 = \omega_2' \\
&\frac{\omega_1}{\omega_3'} = \frac{\omega_3}{\omega_1} = \frac{\omega_5'}{\omega_7} = \frac{\omega_7'}{\omega_5} = \frac{\omega_6'}{\omega_8} = \frac{\omega_8'}{\omega_6} \\
&\frac{\omega_2}{\omega_4} = \frac{\omega_4'}{\omega_2} = \frac{\omega_5}{\omega_8} = \frac{\omega_8'}{\omega_5'} = \frac{\omega_6}{\omega_7} = \frac{\omega_7'}{\omega_6'} \\
&\omega_9 = \omega_9' = \omega_{10} = \omega_{10}' = 0.
\end{aligned} \tag{4}$$

Consider a triangular lattice of M rows where each row has N vertices with cyclic boundary conditions. The number n of down arrows in each row is conserved (Baxter 1969) and the transfer matrix is a block diagonal matrix with one block for each value of $n = 0, 1, \ldots, 2N$. The free energy per vertex is

$$F = -kT \lim_{M,N\to\infty} \frac{1}{MN} \ln Z = -kT \lim_{N\to\infty} \frac{1}{N} \ln \Lambda \tag{5}$$

where Λ is the largest eigenvalue of the transfer matrix. Let us denote by m the number of vertices where two down arrows point out at the same vertex in the lower row. If $\omega_9 = \omega_9' = \omega_{10} = \omega_{10}' = 0$, then each block matrix is reducible and we need to consider only the subspace with definite values of n and m (Lin 1978). The largest eigenvalue in the subspace of n and m is denoted by $\Lambda(n, m)$.

We now consider the model whose vertex weights satisfy the equations (4). It is easy to check that $\Lambda(n, m) = \Lambda(n-2m, 0)$ and therefore we only need to consider the case $m = 0$ (Lin 1978). Our model is not a special case of the model of Sacco and Wu (1975) since the vertex weights do not satisfy the free-fermion conditions. However our model is in some sense equivalent to a special case of their model. To see this, regard their bonds as down- or left-pointing arrows (Lin 1978). If $m = 0$, then their configurations $\bar{f}_0$, $\bar{f}_{14}$, $\bar{f}_{15}$, $\bar{f}_{16}$ do not occur, so these weights can be set equal to zero. Their free-fermion conditions (16) and (17) are then satisfied and

$$\begin{aligned}
&f_0 = \omega_1 \qquad f_{14} = \omega_2 \qquad f_{16} = \omega_3 \qquad f_{15} = \omega_4 \\
&f_{34} = \omega_5 \qquad \bar{f}_{34} = \omega_5' \qquad f_{24} = \omega_6 \qquad \bar{f}_{24} = \omega_6' \\
&f_{25} = \omega_7 \qquad \bar{f}_{25} = \omega_7' \qquad f_{35} = \omega_8 \qquad \bar{f}_{35} = \omega_8'.
\end{aligned} \tag{6}$$

All other f_{ij} and $\bar{f}_{ij}$ are zero. Their $D(\theta, \phi)$ in equation (20) is then given by

$$D(\theta, \phi) = |(\omega_2/\omega_1)\, e^{i\phi} - e^{i(\theta+\phi)} + (\omega_7 + \omega_7')/\omega_1|^2. \tag{7}$$

The free energy is therefore given by their equation (20):

$$-\frac{F}{kT} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln(\omega_7 + \omega_7' + \omega_1 e^{i\alpha} + \omega_2 e^{i\beta})\, d\alpha\, d\beta \tag{8}$$

where $\alpha = \theta + \phi - \pi$ and $\beta = \phi$. The integral (8) reduces to (3) when $\omega_1 = \omega_2 = \omega$ and $\omega_7 = \omega_7' = \tilde{\omega}$. The analytic properties of integral (8) have been discussed in detail by Wu

and Lin (1975). Our model (4) is similar to the model (c) of Wu and Lin. Their result is

$$\begin{aligned} F &= -kT\ln(\omega_7+\omega_7') && \text{if } \omega_7+\omega_7' \geqslant \omega_1+\omega_2 \\ &= \min(e_1, e_2) && \text{if } |\omega_1+\omega_2| \geqslant \omega_7+\omega_7' \\ &= e_1 - \frac{kT}{2\pi}\int_{-\theta_1}^{\theta_1} \mathrm{d}\theta\, \ln[(\omega_7+\omega_7'+\omega_2\,\mathrm{e}^{-\mathrm{i}\theta})/\omega_1] && \text{(otherwise)} \end{aligned} \tag{9}$$

where $|\omega_7+\omega_7'+\omega_2\,\mathrm{e}^{-\mathrm{i}\theta_1}| = \omega_1$. The critical temperature is determined by

$$2\max(\omega_1, \omega_2, \omega_7+\omega_7') = \omega_1+\omega_2+\omega_7+\omega_7'. \tag{10}$$

The specific heat diverges with an exponent $\frac{1}{2}$ above the critical temperature (Wu and Lin 1975).

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